Review Notes for IB Standard Level Math

Contents

1 Algebra

1.1 Rules of Basic Operations

You need to be familiar with the rules of basic operations shown in Table 1.

$a\left(b+c\right)=ab+ac$	$\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$
$\frac{a}{b}+\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{d}+\frac{c}{d}\cdot\frac{b}{b}=\frac{ad+cb}{bd}$	$-\left(a+b\right)=-a-b$
$-\left(a-b\right)=b-a$	$\left(a+b\right)^2=a^2+2ab+b^2$
$\left(a-b\right)^2=a^2-2ab+b^2$	$\left(a+b\right)\left(a-b\right)=a^2-b^2$
$\left(a+b\right)\left(c+d\right)=ac+ad+bc+bd$	

Table 1: Basic Operations

1.2 Rules of Roots

You need to be familiar with the rules of roots shown in Table 2.

$\sqrt{ab}=\sqrt{a}\sqrt{b}$	$\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}$	$a^{\frac{1}{q}}=\sqrt[q]{a}$	$a^{\frac{p}{q}}=\left(\sqrt[q]{a}\right)^p$
$a^{-\frac{p}{q}}=\frac{1}{\left(\sqrt[q]{a}\right)^p}$	$\frac{p}{\sqrt{a}}\cdot\frac{\sqrt{a}}{\sqrt{a}}=\frac{p\sqrt{a}}{a}$	$\left(\sqrt{a}\right)^2=a$	$\left(\sqrt[q]{a}\right)^q=a$

Table 2: Rules of Roots

1.3 Rules of Exponents

You need to be familiar with the rules of exponents shown in Table 3.

$a^0=1$	$a^1=a$
$a^b\cdot a^c=a^{b+c}$	$\left(a^b\right)^c=a^{b\cdot c}$
$\left(\frac{a}{d}\right)^c=\frac{a^c}{d^c}$	$a^{(-c)}=\frac{1}{a^c}$
$\frac{a^b}{a^c}=a^{(b-c)}$	$\frac{a^b}{d^e}\cdot\frac{d^c}{a^f}=\frac{a^b}{a^f}\cdot\frac{d^c}{d^e}=a^{b-f}\cdot d^{c-e}=\frac{a^{b-f}}{d^{e-c}}$

Table 3: Rules of Exponents

1.4 Allowed and Disallowed Calculator Functions During the Exam

The International Baccalaureate documentation for examiners dated 18/11/2015, provides the following guidance about the functions of the TI NSpire calculator during an exam. The following options in "Press to Test" mode must be ticked and therefore **blocked**: *Limit geometry functions*; *Disable function and conic grab and move*; *Disable vector functions including eigenvectors and eigenvalues*; *Disable isPrime function*; *Disable differential equation graphing*; *Disable 3D graphing*; *Disable implicit graphing, conic templates, conic analysis, and geometric conics*. The following options in "Press to Test" mode must be unticked and therefore **allowed**: *Disable inequality graphing*; *Limit trigonometric functions*; $\log_b x$ *template and summation functions*; *Disable Polynomial Root Finder and Simultaneous Equation Solver*.

1.5 Sequences and Series

A *sequence* is an ordered list of numbers. The list can be finite or infinite. We name the sequence with a letter like u, and refer to an individual element (or "term") of this ordered list using a subscript that represents the ordinal position the particular element occupies in the list. The first term in the sequence u is denoted by u_1, the second term by u_2, and so on. The nth term in the sequence is u_n.

You describe a sequence by giving a formula for its nth term. This could be something simple like the sequence each of whose terms is the number 1:

$$u_n = 1$$

More typically, the formula for the nth term involves the variable n in some way. It might be involved in calculating the value of the term like:

$$u_n = n^2 \qquad \text{or} \qquad u_n = \frac{1}{n}$$

The n can also appear in the subscript to reference the values of previous terms. For example, after defining the value of the first and second term, this sequence defines the nth term to be the sum of the previous two terms:

$$\begin{aligned} u_1 &= 1 \\ u_2 &= 1 \\ u_n &= u_{n-1} + u_{n-2} \end{aligned}$$

A *series* is the sum of the terms in a sequence.

1.6 Arithmetic Sequences and Series

A sequence is *arithmetic* if you compute its nth term by adding a constant value d to the previous element:

$$u_n = u_{n-1} + d$$

This value d is known as the *common difference* because if you take any term in the sequence and subtract the value of the preceding one, the difference is d. This is a result of rearranging the previous equation to get:

$$u_n - u_{n-1} = d$$

Given the above definition for u_n involving d, we can also represent the nth element of an arithmetic sequence as the first element to which the common difference d has been added $n-1$ times.

$$u_n = u_1 + (n-1)d$$

An *arithmetic series* is the sum of an arithmetic sequence.

1.7 Sum of Finite Arithmetic Series $(u_1 + \cdots + u_n)$

It only makes sense to talk about the sum of a *finite* arithmetic sequence. Since the elements of an *infinite* arithmetic sequence continue growing by the common difference, we don't try to compute the sum of the corresponding infinite arithmetic series because its value would always be ∞ (or $-\infty$ if d is a negative number).

To calculate the sum of the first n terms of a *finite* arithmetic sequence we use the formula:

$$S_n = \frac{n}{2}\left(u_1 + u_n\right)$$

This formula comes from observing that the sum of the pair of $u_1 + u_n$ (the first and last terms) is the same as the sum of $u_2 + u_{n-1}$ (the second and penultimate terms), which is the same as the sum of $u_3 + u_{n-2}$ (the third and antepenultimate terms), etc. Since we're pairing up the terms, we end up with $\frac{n}{2}$ pairs having the same sum as $u_1 + u_n$. By multiplying this sum by the number of pairs, we get the sum of all of the terms of the series up to n. Since above we saw that $u_n = u_1 + (n-1)d$, we can also write this formula as:

$$S_n = \frac{n}{2}\left(u_1 + (u_1 + (n-1)d)\right) = \frac{n}{2}\left(2u_1 + (n-1)d\right)$$

1.8 Partial Sum of Finite Arithmetic Series $(u_j + \cdots + u_n)$

Sometimes you might need to calculate the sum of a series starting at a jth term other than the first term, that is where $j > 1$. These kind of problems are easy once you understand that the sum of the first n terms equals the sum of the first $j-1$ terms plus the sum of the terms from the jth position to the nth. In other words,

$$S_n = (u_1 + \cdots + u_{j-1}) + (u_j + \cdots + u_n)$$
$$S_n = S_{j-1} + S_{j\ldots n}$$

Rearranging the equation we have:

$$S_{j\ldots n} = S_n - S_{j-1}$$

Therefore, to find the sum of the series from the jth term through the nth term (inclusive), calculate S_n and subtract from it the value of S_{j-1}.

1.9 Geometric Sequences and Series

A sequence is *geometric* if you compute its nth term by multiplying the previous element by a constant value r:

$$u_n = u_{n-1} \cdot r$$

This value r is known as the *common ratio* because if you take any term in the sequence and divide by the value of the preceding one, the ratio is r. This is a result of rearranging the previous equation to get:

$$\frac{u_n}{u_{n-1}} = r$$

Given the above definition for u_n involving r, we can also represent the nth element of an arithmetic sequence as the first element by which the common ratio r has been multiplied $n-1$ times.

$$u_n = u_1 \cdot r^{n-1}$$

A *geometric series* is the sum of a geometric sequence.

1.10 Sum of Finite Geometric Series

To calculate the sum of the first n terms of a geometric series, use the formula:

$$S_n = \frac{u_1 (r^n - 1)}{r - 1}, \quad r \neq 1$$

1.11 Sum of Infinite Geometric Series

If the absolute value of the common ratio is less than one, then each time we compute the next term of a geometric sequence it is some fraction of the previous term. In other words, the new term is smaller in absolute value than the previous term. Another way to think of this is that the new term is closer to zero than the previous term was. This means we can calculate the sum of the *infinite* geometric series because the terms tend to zero as n gets larger and larger.

To calculate the sum of an infinite geometric series, use the formula:

$$S_\infty = \frac{u_1}{1 - r}, \quad |r| < 1$$

1.11.1 Example Involving Sum of Infinite Geometric Series

As shown in Figure 1, the sides of an outermost square are 16 units. To create a smaller square, connect midpoints of the original square. Consider the sequence A_n of the areas of the shaded triangles in the lower-left corner of each square, formed when the next smaller square is drawn. Find the sum of the areas of these shaded triangles when this process continues infinitely.

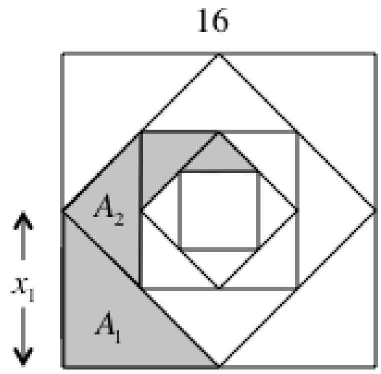

Figure 1: Progressively Smaller Triangle Areas

Since the original side length is 16, the legs of the first shaded triangle are 8, and the area $A_1 = \frac{1}{2} \cdot 8 \cdot 8 = 32$. Using Pythagorean Theorem, we can see that for a square of side-length a, its hypotenuse will always be $\sqrt{a^2 + a^2} = a\sqrt{2}$, so the hypotenuse of the first triangle is $8\sqrt{2}$. This means that the legs of the second triangle are half of this, or $4\sqrt{2}$. Therefore the area of the second triangle is $A_2 = \frac{1}{2} \cdot 4\sqrt{2} \cdot 4\sqrt{2} = 16$. The hypotenuse of the second shaded triangle is $4\sqrt{2}\sqrt{2} = 8$, so the legs of the third triangle are half of this: 4. Hence, the area of the third shaded triangle is $A_3 = \frac{1}{2} \cdot 4 \cdot 4 = 8$. The hypotenuse of the third shaded triangle is $4\sqrt{2}$, so the legs of the third triangle are half of this: $2\sqrt{2}$. So, the area of the fourth shaded triangle is $A_4 = \frac{1}{2} \cdot 2\sqrt{2} \cdot 2\sqrt{2} = 4$. So, the pattern that is emerging is that $A = \{32, 16, 8, 4, \ldots, 2^{6-n}, \ldots\}$.

Considering the sequence A of the triangle areas, we see that each subsequent term is obtained by multiplying the previous term by $\frac{1}{2}$, so the sequence of the areas is a geometric sequence with common ratio $r = \frac{1}{2}$. Since the common ratio is less than one in absolute value, it makes sense to compute the inifite sum, which is given by the formula in Section 1.11, “Sum of Infinite Geometric Series”:

$$\sum_{n=1}^{\infty} A_n = \frac{A_1}{1-r} = \frac{32}{1-\frac{1}{2}} = 64$$

1.12 Sigma Notation

Sigma notation is another convenient way to represent a series. Instead of saying an English phrase like “the sum of the first 5 terms of the sequence defined by $u_n = 2n^2 + 1$”, we can write the same expression using the much more compact formula:

$$\sum_{n=1}^{5} 2n^2 + 1$$

The $\sum$ symbol is the capital Greek letter sigma, the first letter in the word “sum”. The $n = 1$ below the $\sum$ indicates the variable whose value will be changing to compute the terms in the series, and it also indicates the starting value of this variable. The 5 above the $\sum$ symbol provides the ending value of the variable n specified below the symbol. You compute each term in the sequence by substituting the current value of the changing variable into the expression the follows the $\sum$ symbol. The series is the sum of these terms.

In the example above, the variable is n and its value starts at 1 and goes to 5, incrementing by 1 each time. This means that we can “expand” the compact sigma notation in the example into this expression:

$$\begin{aligned}
\sum_{n=1}^{5} 2n^2 + 1 &= \left(\underbrace{2\,(\mathbf{1})^2 + 1}_{n=1}\right) + \left(\underbrace{2\,(\mathbf{2})^2 + 1}_{n=2}\right) + \left(\underbrace{2\,(\mathbf{3})^2 + 1}_{n=3}\right) + \left(\underbrace{2\,(\mathbf{4})^2 + 1}_{n=4}\right) + \left(\underbrace{2\,(\mathbf{5})^2 + 1}_{n=5}\right) \\
&= (2 \cdot 1 + 1) + (2 \cdot 4 + 1) + (2 \cdot 9 + 1) + (2 \cdot 16 + 1) + (2 \cdot 25 + 1) \\
&= (3) + (9) + (19) + (33) + (51) \\
&= 115
\end{aligned}$$

In the example above, by computing the difference between two pairs of consecutive terms, we can observe that the sequence (and hence the series) is not arithmetic because the terms do not share a common difference:

$$\begin{aligned}
u_2 - u_1 &= 9 - 3 = 6 \\
u_3 - u_2 &= 19 - 9 = 10 \neq 6
\end{aligned}$$

By computing the ratio between two pairs of consecutive terms, we can also observe that the sequence (and hence the series) is not geometric because the terms do not share a common ratio:

$$\begin{aligned}
\frac{u_2}{u_1} &= \frac{9}{3} = 3 \\
\frac{u_3}{u_2} &= \frac{19}{9} \neq 3
\end{aligned}$$

Of course there is nothing special about the letter n and the value of the variable being incremented from the start index to the end index need not start with the value 1. The sigma notation can be used with any letter for the variable and any integer values for the index start